GAMES-THEORETICAL ENCOUNTER OF MOTIONS WITH BOUNDED IMPULSES

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The problem of the games-theoretical encounter of similar linear objects under the condition of minimax of the time T until coincidence of the specified phase vector coordinates and under restrictions on the controlling force impulses [1 and 2] is considered. A computational scheme modifying the extremal aiming rule for the case in question is justified.

1. Formulation of the problem. Let us consider the problem [3] of the minimax of the time T until encounter of the pursuing ($y[t]$) and pursued ($z[t]$) motions described by Eqs.

$$\frac{dy}{dt} = Ay + Bu, \qquad \frac{dz}{dt} = Az + Bv \tag{1.1}$$

where only the controls u and v are allowed. The realizations $u[t]$ and $v[t]$ of these controls satisfy the integral restrictions

$$\int_\tau^\infty \|u[t]\| dt \leqslant \mu[\tau], \qquad \int_\tau^\infty \|v[t]\| dt \leqslant \nu[\tau] \tag{1.2}$$

which are interpreted as restrictions on the impulses of the controlling forces. Here y, z are the phase n-vectors of the objects; u, v are the r-vectors of the controls; the symbol $\|q\|$ denotes the Euclidean norm of the vector q. The vectors in question are regarded as vector columns; the superscript $*$ denotes transposition, and the symbol $\{Q\}_{[m]}$ represents the matrix of the first m rows of the matrix Q.

The pursuit goal is coincidence of the vectors $\{y[t]\}_{[m]}$ and $\{z[t]\}_{[m]}$, where m is a given number ($m \leq n$). The control u is formed by the feedback principle at each instant $t = \tau$ on the basis of the realized values $y[\tau]$, $z[t]$, $\mu[\tau]$, and $\nu[\tau]$, i.e.

$$u[\tau] = u[y[\tau], z[\tau], \mu[\tau], \nu[\tau]] \tag{1.3}$$

In order to distinguish the program controls u and v specified in advance in the form of functions of the time t from the realizations of the controls u and v constructed by the feedback principle in the form of the functions

$$u = u[y, z; \mu, \nu], \qquad v = v[y, z; \mu, \nu] \tag{1.4}$$

which, however, are realized in each specific case in the closed system as functions of the time t. We shall denote the former by the symbols $u(t)$ and $v(t)$ and the latter by $u[t]$ and $v[t]$. In general, the square brackets containing the arguments t and τ will indicate that we are referring to the realizations of the processes in question. As our permissible controls u and v we shall consider controls whose realizations $u[t]$ and $v[t]$ can be represented as the sum of a bounded integrable function and a linear combination of δ-functions $\delta(t - t_*)$.

Thus, we are required to find the control u (1.4) which ensures that

$$T^\circ = \min_u \sup_v T \quad \text{for} \quad \{y[\tau + T]\}_{[m]} = \{z[\tau + T]\}_{[m]} \tag{1.5}$$

regardless of the initial conditions $y[\tau]$, $z[\tau]$, $\mu[\tau]$, $\nu[\tau]$ from their prescribed range. The control u must be constructed in the form (1.4); the control v can be chosen either from among functions (1.4) or from among the program controls $v(t)$.

N o t e 1.1. The present problem is one of the theory of differential games. (see the bibliography in survey [5]). Under restrictions (1.2) it has the distinctive feature of making control realizations in the form of pulse δ-functions expedient. This creates certain difficulties in direct solution of the problem [2]. In the case $m = n = 2$ the problem is solved in great detail [1] with allowance for the difficulties just mentioned. We shall describe a modification of problem (1.1), (1.2), (1.5) for the general case $n \geq m \geq 1$ and propose a scheme for its solution. This modification is similar to the pursuit problem for linear objects (1.1) described in [4], but is subject to the restrictions

$$\| u[t] \| \leqslant \mu, \quad \| v[t] \| \leqslant \nu \tag{1.6}$$

The scheme which we shall propose enables us to circumvent certain of the difficulties and to make use of the extremal aiming rule [4] in somewhat altered form. We note that we are concerned here only with the m i n i m a x of the time T until encounter. The problem of the saddle point of the game where max min T = min max T will not be discussed here as it is by the author of [1].

Under integral restrictions on u and v the distinction between the two problems is significant. This is because under restrictions (1.6) investigation of the maximin of T often presents no further difficulties, while in the case of integral restrictions on the controls u and v the problem of the maximin of the time T until encounter often requires fresh and specific investigation (e.g. see [6]).

2. Modification of the problem. The modification we are about to describe is based on conversion to a discrete system followed by taking of a limit. A regularizing scheme of this type, which is well suited for simulation on an electronic computer, is described in [2] for a particular case of problem (1.1), (1.2). We shall describe its construction for the general case of problem (1.1), (1.2).

Let the pursuit process begin at the instant $t = t_0$. We introduce the sequence $\{\tau_k\}$ ($k = 0, 1, \ldots$) of instants $t = \tau_k$ ($\tau_0 = t_0$, $\tau_{k+1} - \tau_k = \Delta > 0$) and assume that the choice of the control $u[t]$ over the entire interval $[\tau_k, \tau_{k+1})$ is determined by the realized values $y[\tau_k]$, $z[\tau_k]$, $\mu[\tau_k]$, $\nu[\tau_k]$. To the arguments which determine the function $u[t]$ for $\tau_k \leq t < \tau_{k+1}$ we add the variable $\vartheta[\tau_k]$ whose meaning will be made clear below. For the present we note that when $\tau_k > \tau_0$ the quantity $\vartheta[\tau_k]$ is determined on the basis of the values $y[\tau_k]$, $z[\tau_k]$, $\mu[\tau_k]$, $\nu[\tau_k]$ and $\vartheta[\tau_{k-1}]$; when $\tau_k = \tau_0$, i.e. at the initial instant of pursuit, the quantity $\vartheta[\tau_0]$ is determined on the basis of the values $y[\tau_0]$, $z[\tau_0]$, $\mu[\tau_0]$, $\nu[\tau_0]$.

Thus, let us assume that we have chosen some algorithm which determines the control u from the rule

$$u[t] = u_t\,[y[\tau_k],\ z[\tau_k],\ \mu[\tau_k],\ \nu[\tau_k],\ \vartheta[\tau_{k-1}]] \qquad (\tau_k \leqslant t < \tau_{k+1}) \tag{2.1}$$

This algorithm, which includes a description of the method of computing functions (2.1) for each sufficiently small $\Delta > 0$, will be referred to for brevity as the "control law", or still more briefly, as the "control" u.

Let us denote by $t = \tau + T^{\varepsilon}_{u,v}$ the instant at which the inequality

$$\| \{y[t] - z[t]\}_{[m]} \| \leqslant \varepsilon \qquad (\varepsilon > 0) \tag{2.2}$$

is first fulfilled under the chosen control laws u and v.

Here τ is some temporarily fixed instant $t = \tau \geq t_0$. (Since the pulse controls $u[t]$ and $v[t](v(t))$ are generally permitted, the quantities $y[t]$ and $z[t]$ in (2.2) should be interpreted strictly, as the quantities $y[t+0]$ and $z[t+0]$. This remark should be borne in mind in similar cases below). The result of pursuit under the chosen control law (2.1) can be estimated by means of the quantity

$$\gamma_u = \sup_\varepsilon\,[\limsup_{\Delta \to 0}\,(\sup_v T^{\varepsilon}_{u,v})] \qquad (\varepsilon > 0) \tag{2.3}$$

Our task is to select a control $u_t[y, z, \mu, \nu, \vartheta]$ (2.1) which minimizes the quantity γ_u

for each possible state $y[\tau]$, $z[\tau]$, $\mu[\tau]$, $\nu[\tau]$, $\tau \geq \tau_0$, of objects (1.1).

We must therefore find the optimal control

$$u[t] = u^\circ_t[y, z, \mu, \nu, \vartheta]$$

which yields the minimum

$$T^\circ = \Upsilon_{u^\circ} = \min_u \Upsilon_u \tag{2.4}$$

In other words, the optimal control law u° must have the following property: whatever the number $\varepsilon > 0$, there exists a number $\Delta = \Delta_0$ such that when $\Delta \leq \Delta_0$ the control $u = u_t^\circ[y, z, \mu, \nu, \vartheta]$ realized in the form (2.1) satisfies condition (1.2) and ensures the ε-convergence (2.2) of the motions $y[\tau]$ and $z[\tau]$ not later than at the instant

$$t \leqslant \tau + T^\circ + \varepsilon$$

regardless of the permissible control $v[t]$ or $v(t)$ which satisfies restriction (1.2).

There must not exist a control $u = u_*$ which would ensure fulfillment of analogous conditions for $T_* < T^\circ$.

N o t e 2.1. Introduction of the argument $\vartheta[\tau_k]$ determined from the values $\vartheta[\tau_{k-1}]$ introduces a certain after effect into the control law. This is undesirable and can be excluded by discriminating [2] the motion $z[t]$, i.e. by allowing the use of the realized values of $v[t]$ or at least of $v[t-\eta]$ ($\eta > 0$ is a small lag) in computing the control $u[t]$ at each instant t. This simplifies our problem considerably, but deprives it of the character of a positional game to a probably greater degree than does the above scheme including the quantity $\vartheta[\tau_k]$. Moreover, introduction of the quantity $\vartheta[\tau_k]$ instead of $v[t]$ as an argument of the control law u has the advantage that the quantity $\vartheta[\tau_k]$ is computed from the instantaneous positional quantities $y[t]$, $z[t]$, $\mu[t]$, and $\nu[t]$ stably, whereas determination of the controlling force $v[t]$ from the changes in $z[t]$ and $\nu[t]$ sometimes involves substantial errors.

3. Solution of the modified problem. We begin by considering two ancillary problems of optimal program control.

P r o b l e m 1. Let us consider the controlled system

$$dx/dt = Ax + Bw \tag{3.1}$$

For given $\varepsilon \geq 0$, $\zeta \geq 0$ and initial conditions τ, $x[\tau]$ we are to determine the optimal control $w^\varepsilon_{x[\tau],\zeta}(t)$ $(t \geq \tau)$ which is restricted by the condition

$$\int_\tau^\infty \|w(t)\| dt \leqslant \zeta \tag{3.2}$$

and ensures the most rapid attainment by system (3.1) of the state

$$\|\{x(\tau + T)\}_{[m]}\| \leqslant \varepsilon \tag{3.3}$$

We shall denote the time-optimal operating period for Problem 1 by the symbol $T_\varepsilon[x[\tau], \zeta]$.

P r o b l e m 2. Let the numbers $T > 0$ and $\zeta \geq 0$ be given. Under the given initial conditions τ, $x[\tau]$ we are to find the optimal control $w_{x[\tau],\zeta}(t)^\circ$ $(t \geq \tau)$ which is restricted by condition (3.2) and ensures the minimum

$$\varepsilon^\circ = \min \|\{x(\tau + T)\}_{[m]}\| \tag{3.4}$$

Let us construct the control u° on the basis of the solutions of Problems 1 and 2. We assume that the inequality $\mu[\tau_0] \geq \nu[\tau_0]$ is fulfilled for $\tau = \tau_0$, and that when

$$x[\tau_0] = y[\tau_0] - z[\tau_0], \qquad \zeta = \zeta[\tau_0] = \mu[\tau_0] - \nu[\tau_0]$$

Problem 1 for $\varepsilon = 0$ has the finite solution $T = T_0[\tau_0]$. This condition will be considered fulfilled in the discussion which follows. The solution of Problem 1 is known [7 to 9]. The optimal control $w^\circ(t)$ consists of a sequence of pulses and is given by

$$w^0_{x[\tau_0],\,\zeta[\tau_0]}(t) = \sum_{s=1}^{l} \lambda_s \delta(t - t_s) \tag{3.5}$$

Let the interval $[\tau_0, \tau_1)$ contain the points $t_1, \ldots, t_p$. We set $\vartheta[\tau_0] = \tau_0 + T_0[\tau_0]$ and construct the function $u[t]$ from Formula

$$u[t] = u_t^\circ[x[\tau_0], \zeta[\tau_0], \vartheta[\tau_0]] = \frac{1}{\Delta}\sum_{s=1}^{p}\lambda_s \quad (\tau_0 \leqslant t < \tau_1) \tag{3.6}$$

(If the interval $[\tau_0, \tau_1)$ does not contain any point t_s, we set $u[t] \equiv 0$ for $\tau_0 \leq t < \tau_1$. An analogous remark should be borne in mind in all similar situations to follow, although this will not be noted for the sake of brevity.)

Now let us consider the instant $t = \tau_k > \tau_0$ and assume that the quantity $\vartheta[t_{k-1}]$ is known. If $\varepsilon = 0$ and if under the conditions

$$\tau = \tau_k, \; x[\tau_k] = y[\tau_k] - z[\tau_k], \; \zeta[\tau_k] = \mu[\tau_k] - \nu[\tau_k]$$

Problem 1 has the solution

$$T_0[\tau_k] \leqslant \vartheta[\tau_{k-1}] - \tau_k,$$

then, setting $\vartheta[\tau_k] = \tau_k + T_0[\tau_k]$, we can take the corresponding solution

$$\overset{\circ}{w}_{x[\tau_k], \zeta[\tau_k]}(t) = \sum_{s=1}^{l^{(k)}} \lambda_s^{(k)} \delta(t - t_s^{(k)}) \tag{3.7}$$

and use it to construct again the function u° in the form

$$u^\circ[t] = u_t^\circ[x[\tau_k], \zeta[\tau_k], \vartheta[\tau_k]] = \frac{1}{\Delta}\sum_{s=1}^{p^{(k)}} \lambda_s^{(k)} \quad (\tau_k \leqslant t < \tau_{k+1}) \tag{3.8}$$

Here $p^{(k)}$ denotes the number of points $t_s^{(k)}$ from (3.7) which enter into the interval $[\tau_k, \tau_{k+1})$.

On the other hand, if Problem 1 does not have a solution $T_0[\tau_k] \leq \vartheta[\tau_{k-1}] - \tau_k$ for $\varepsilon = 0$ and for the given conditions $\tau = \tau_k$, $x[\tau_k]$, $\zeta[\tau_k]$, we must solve Problem 2 for the given $\tau = \tau_k$, $x[\tau_k]$, $\zeta[\tau_k]$ and $T = \vartheta[\tau_{k-1}] - \tau_k$.

Let the solution of this problem be $\varepsilon^\circ = \varepsilon^\circ[\tau_k]$. Having found the number $\varepsilon^\circ[\tau_k]$, we solve Problem 1 for $\varepsilon = \varepsilon^\circ[\tau_k]$. By our choice of ε, Problem 1 has the solution $T_{\varepsilon^\circ}[\tau_k] \leq \vartheta[\tau_{k-1}] - \tau_k$. We now set $\vartheta[\tau_k] = \tau_k + T_{\varepsilon^\circ}[\tau_k]$. The solution $\overset{\varepsilon^\circ}{w}_{x[\tau_k], \zeta[\tau_k]}(t)$ of Problem 1 is again of the form (3.7), and the function $u[t] = u_t^\circ$ is constructed from this solution again in the form (3.8). This construction is fulfilled as long as $\vartheta[\tau_{k-1}] > \tau_k$. If $\vartheta[\tau_{k-1}] \leq \tau_k$ at some instant τ_k, then from that instant we always set $\vartheta[\tau_k] = \tau_k$; all the other constructions which determine the function u_t° remain unchanged.

The resulting control u_t° solves the problem posed in Section 2.

Let us give a brief justification of this statement. First, let $\tau = \tau_0$. Choosing a control $v[t] = \mu[t]\, u[t]/\nu[t]$ for each $u[t]$, we can verify that $T^\circ[\tau_0]$ is not smaller than $T_0[\tau_0]$, since system (3.1) would otherwise be brought by some control $w(t) = u[t] - v[t]$ satisfying condition (3.2) from the given state $x[\tau_0]$ to the state $\{x(\tau_0 + T_*)\}_{[m]} = 0$ for $T_* < T_0$. Thus, in order to prove the statement in the case $\tau = \tau_0$ it is enough to verify that for each $\varepsilon > 0$ chosen for sufficiently small values of $\Delta > 0$, the constructed control u° ensures ε-convergence (2.2) of the motions $y[t]$ and $z[t]$ by the instant $t \leq \tau_0 + T_0[\tau_0] + \varepsilon$ regardless of the character of the permissible control v. Let us verify this.

First, we note from the construction of the function u_t° that the quantity $\zeta[\tau_k]$ is always nonnegative. Hence, construction of $\vartheta[\tau_k]$ and u_t° $(\tau_k \leq t < \tau_{k+1})$ is possible during the entire time until the required convergence of $y[t]$ and $z[t]$ (or over an infinite time if this convergence does not take place). By construction, the values of $\vartheta[\tau_k]$ do not increase until $\vartheta[\tau_k] > \tau_{k+1}$. Hence, either the required ε-convergence occurs for $t \leq \vartheta[\tau_k] \leq \tau_0 + T_0[\tau_0]$ as required, or there arrives an instant $\tau_{k'}$ when Eq. $\tau_{k'} = \vartheta[\tau_{k'}]$ is fulfilled, although the required ε-convergence will not yet have occurred by this time.

Let us consider the second possibility. We can verify that for sufficient small values of Δ, the quantities $\varepsilon^\circ[\tau_k]$ $(k \leq k')$ which occur in the course of solution of ancillary Problem 2 will not exceed a number $\eta > 0$ chosen in advance. In fact, $\varepsilon^\circ[\tau_0] = 0$. On the other hand, the possible increase in the quantity $\varepsilon^{\circ}[\tau_k] \to \varepsilon^{\circ}[\tau_{k+1}]$ in a single interval can be estimated as follows.

Let the control $v_*[t]$ operate over the interval $\tau_k \leq t < \tau_{k+1}$, and let an impulse characterized by the quantities

$$\varkappa = \int_{\tau_k}^{\tau_{k+1}} v_*[t]\,dt, \qquad \varkappa_* = \int_{\tau_k}^{\tau_{k+1}} \| v_*[t] \| \,dt$$

be generated over this interval.

If the control operating in the same interval were

$$u_*[t] = \sum_{s=1}^{p(k)} \lambda_s^{(k)} \delta(t - t_s^{(k)}) + v_*[t]$$

then by virtue of the optimality of the control constituting the first term we would have the inequality $\varepsilon_*{}^{\circ}[\tau_{k+1}] \leq \varepsilon^{\circ}[\tau_k]$. This is because optimal control (3.7) would then be operating in system (3.1), where $x = y - z$. Let us denote by $x_*[\tau_{k+1}]$ the value of the difference $y - z$ which would be realized under the controls u_*, v_*. However, the interval $[\tau_k, \tau_{k+1})$ is in fact associated with a control u_t° of the form (3.8). The control $u_{**}[t] = u_t^{\circ} + \varkappa\delta^-(t - \tau_{k+1})$, operating in system (1.1) with the control $v_*[t]$, would bring the system to the state $x_{**}[\tau_{k+1}] = y - z$, which differs from $x_*[\tau_{k+1}]$ by an amount on the order of

$$(\varkappa_* + \sum_{s=1}^{p(k)} |\lambda_s^{(k)}|)\,\Delta$$

Here the symbol $\delta^-(t)$ denotes the "left-hand" δ-function which generates the impulse at the point $t = -0$. From this we infer that the quantity $\varepsilon_{**}{}^{\circ}[\tau_{k+1}]$ which would result in this case satisfies the inequality

$$\varepsilon_{**}{}^{\circ}[\tau_{k+1}] \leqslant \varepsilon_*{}^{\circ}[\tau_k] + \lambda\Delta\left(\varkappa_* + \sum_{s=1}^{p(k)} |\lambda_s^{(k)}|\right) \qquad (\lambda = \text{const})$$

But the significance of the quantity $\varepsilon^{\circ}[\tau_{k+1}]$ which arises in reality implies that $\varepsilon^{\circ}[\tau_{k+1}] \leq \varepsilon_{**}{}^{\circ}[\tau_{k+1}]$. Thus, we obtain the estimate

$$\xi_k = \varepsilon^{\circ}[\tau_{k+1}] - \varepsilon^{\circ}[\tau_k] \leqslant \lambda\Delta\left(\varkappa_* + \sum_{s=1}^{p(k)} |\lambda_s^{(k)}|\right) \tag{3.9}$$

which in turn implies the estimate

$$\varepsilon^{\circ}[\tau_k] \leqslant \sum_{s=0}^{k} \xi_k \leqslant \lambda\Delta(\mu[\tau_0] + \nu[\tau_0]) = \eta \tag{3.10}$$

From (3.10) we infer, in turn, that if the required convergence of the motions y and z has not occurred by the instant $t = \tau_k'$ when for the first time $\vartheta[\tau_k] = \tau_k'$, then in any case the accessibility domain (see [2]) $G_2[\tau_k', \tau_k', z[\tau_k']]$ of the motion $z[t]$ lies in the η-neighborhood of the accessibility domain $G_1[\tau_k', \tau_k', y[\tau_k']]$ of the motion $y[t]$.

By virtue of the arbitrary smallness of the quantity η (3.10) we conclude from this that for sufficiently small Δ the required ε-convergence of the motions $y[t]$ and $z[t]$ occurs not later than $2\nu[\tau_0]/\varepsilon$ steps after the instant $\tau = \tau_k'$. This completes our verification of the optimality of the control u_t° constructed for the chosen initial data $\tau = \tau_0$ and $x[\tau_0]$.

Optimality of the control u_t° which ensures minimax ε-convergence of the motions $y[t]$ and $z[t]$ (in the sense of (2.3) and (2.4)) when the quantity $\gamma_u{}^{\circ} = T_0[\tau_k]$ is measured from an arbitrary instant and when the realizations $y[\tau_k]$, $z[\tau_k]$, $\mu[\tau_k]$, $\nu[\tau_k]$ are arbitrary can be proved in the same way. This is because the foregoing statements imply that for sufficiently small values of $\Delta > 0$ the quantity $\varepsilon[\tau_k]$, which determines the entire subsequent course of pursuit, is sufficiently small.

This way we can prove the following statement. Let us be given an arbitrarily small number $\varepsilon > 0$. We can then choose an arbitrarily small Δ_0 $(\Delta \leq \Delta_0)$ such that the following condition is fulfilled for the realization corresponding to the discrete control scheme: if $\tau = \tau_k \geq \tau_0$ is some instant of the control process and if $x[\tau_k] = y[\tau_k] - z[\tau_k]$, $\zeta[\tau_k] = \mu[\tau_k] - \nu[\tau_k]$ are the quantities realized at this instant, then the control u_t° operating for $t \geq \tau_k$ for any permissible control v ensures ε-convergence of the motions $y[t]$ and $z[t]$ not later than at the instant $t \leq \vartheta[\tau_k] + \varepsilon \leq \tau_k + T_0[\tau_k] + \varepsilon$. At the same time, there is no way of choosing the control u_* which, beginning to operate at the instant $t = \tau_k$ for any $\varepsilon > 0$ and for any permissible control v would ensure ε-convergence (2.1) of the motions y and z (1.1) at the instant $t \leq \tau_k + T_* + \varepsilon$, where $T_* < T^\circ$. This implies the optimality of the constructed control u_t° in the sense of conditions (2.2) and (2.3).

Note 3.1. The construction of the control u° described above is based on the solutions of Problems 1 and 2 which must be found for each $\tau = \tau_k$ in the course of the pursuit process. Methods of constructing these solutions are known from the general theory of linear object control. We shall cite them here for completeness, interpreting the problems in question [10] as moment problems. Let $s(t)$ be the solution of Eq.

$$ds/dt = -A^* s \tag{3.11}$$

The realizability condition for the transfer of system (3.1) from the given state $x[\tau]$ to to the state $\{x(\tau + T)\}_{[m]} = \{z\}_{[m]}$ under restriction (3.2) if the problem is interpreted as a moment problem [11], is

$$\{c_T(z)\}_{[m]} = \int\limits_{\tau}^{\tau+T} \{X[T+\tau-t]\, Bw(t)\}_{[m]}\, dt \tag{3.12}$$

where $X(t)$ is the fundamental matrix of solutions of system (3.1) (for $u \equiv 0$) and $c_T(z) = z - X(T)\, x[\tau]$ can be written as

$$\xi_T(l, z) - \zeta\rho_T(l) \leqslant 0 \tag{3.13}$$

for all l.

Here

$$\xi_T(l, z) = l^* c_T(z), \qquad \rho_T(l) = \max_t \| s^*(t)\, B \| \tag{3.14}$$

where l is the vector of the boundary conditions $l = s\ (\tau = T)$ of the solution $s(t) = X[T + \tau - t]l$ of Eq. (3.11), where $l_i = 0$ for $i = m+1, \ldots, n$. Hence, the solution T° of Problem 1 for $\varepsilon = 0$ is defined as the smallest number $T \geq 0$ which satisfies condition (3.13) for $z = 0$. Considering the coordinates x_i $(i = 1, \ldots, m)$ as controlled, i.e. that $\rho(l) > 0$ for $l \neq 0$, then condition (3.13) can be written as

$$\xi^\circ = \xi(l^\circ, 0) = \max_l \xi(l, 0) \leqslant \zeta \qquad \text{for } \rho(l) = 1 \tag{3.15}$$

The optimal control $w^\circ(t)$ itself can be determined from the maximum condition

$$\int\limits_{\tau}^{\tau+T} s^{\circ *}(t)\, Bw^\circ(t)\, dt = \max_w \qquad \text{for} \qquad \int\limits_{\tau}^{\tau+T} \| w(t) \|\, dt \leqslant \xi^\circ \tag{3.16}$$

where $s^\circ[\tau + T] = l^\circ$ is the solution of problem (3.15). For $\varepsilon > 0$ the solution of Problems 1 and 2 follows from the conditions of separation of the accessibility domain of process (3.1) and the sphere $\|z\| \leq \varepsilon$ by the instant $t = \tau + T$ (e.g. see [12 and 13]). At the same time, as is noted in [10], the solutions of Problems 1 and 2 also follow directly from relation (3.13).

To show this it is sufficient, for example, to write (3.13) in the form $\max_l [\xi_T(l, z) - \zeta\rho_T(l)] \leq 0$ for $\|l\| \leq 1$. The condition of entry into the ε-sphere $\|z\| \leq \varepsilon$ takes the form $\min_z \max_l [\xi_T(l, z) - \zeta\rho_T(l)] \leq 0$ for $\|l\| \leq 1$ and $\|z\| \leq \varepsilon$. By virtue of the permutability of the operations min and max in this case, and with allowance for the expressions for ξ and c_T, we obtain from this the condition

$$\max_l [l^* X(T)\, x[\tau] - \zeta\rho_T(l) - \varepsilon] \leqslant 0 \qquad \text{for } \|l\| = 1$$

which determines the solution of Problems 1 and 2 (the smallest $T \geq 0$ for the given ε and conversely, the smallest $\varepsilon \geq 0$ for a given T). The optimal controls $w(t)$ themselves are again determined from maximum conditions similar to (3.16).

N o t e 3.2. The example

$$A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B=\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \zeta[\tau_0]=1, \qquad x[\tau_0]=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

indicates that without introduction of the artifically fixed quantity $\vartheta[\tau_k]$ similar construction of the control u_e based only on the solution of ancillary Problem 1, i.e. on the choice of $\vartheta[\tau_k] = \tau_k + T_0[\tau_k]$ involves difficulties due to the possible loss of the given unstable root $\vartheta[\tau_k]$.

N o t e 3.3. Similar construction of a scheme for regularizing the optimal control u° can also be effected (with appropriate alterations) for the problem of the minimax of the time $T^{(\varepsilon^\circ)}$ until the ε°-encounter $\|\{y[t] - z[t]\}_{[m]}\| \leq \varepsilon^\circ$ of the motions $y[t]$ and $z[t]$ (for a given ε°). Quite naturally, it can also be effected for the more regular problem of the minimax of the quantity $\|\{y(\vartheta) - z(\vartheta)\}_{[m]}\|$ at the given fixed instant $t = \vartheta$ of termination of the process. Finally, the system can be adapted without significant alterations for the case where restrictions (1.2) involve not a Euclidean, but some other norm of the vectors u and v.

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